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Approximation of Solutions of Nonlinear Operator Equation on the Half Line

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Abstract—This paper discusses the solvability and approximation of operator equations on the half line.

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1. INTRODUCTION

In this paper, we discuss existence and approximation for the nonlinear operator equation on the half line

$$y(t) = F\,y(t), \qquad \text{on } [0,\infty). \tag{1.1}$$

Solutions will be sought in $C([0,\infty),\mathbf{R}^k), k \in N^+ = \{1,2,\dots\}$. A particular example of (1.1) will be the nonlinear integral equation

$$y(t) = h(t) + \int_0^\infty \kappa(t,s,y(s))\,ds, \qquad \text{for } t \in [0,\infty). \tag{1.2}$$

Finite section approximations for (1.2) are given by

$$y(t) = h(t) + \int_0^n \kappa(t,s,y(s))\,ds, \qquad \text{for } t \in [0,\infty), \tag*{(1.3)^n}$$

for $n \in N^+$. Note that $(1.3)^n$, for fixed $n \in N^+$, determines $y(t)$ for $t > n$ in terms of $y(x)$ for $x \in [0,n]$ so in fact the finite section approximations reduce to integral equations on bounded intervals (we note as well that various discretization techniques, such as numerical integration, are available for the approximate solution of $(1.3)^n$, $n \in N^+$ fixed). The technique which we present in this paper is to establish existence and approximation of solutions to (1.2) (or more generally (1.1)) which involves using a new fixed point approach for equations on the half line (see [1–4]) together with the well-known notion of strict convergence (see [5–7]).

In particular, we put conditions on κ so that $(1.3)^n$ has a solution x_n for each $n \in N^+$. Then we let $n \to \infty$. Using the notions of collectively compact operators and strict convergence (see [5,6,8]) we will show under reasonable conditions that there exists a subsequence S of N^+ and a $x_0 \in C([0,\infty),\mathbf{R}^k)$ with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,\infty),\mathbf{R}^k)$ and x_0 is a solution of (1.2). In addition under reasonable assumptions we will show that the solution sets of $(1.3)^n$ converge (under an appropriate sense) to the solution set of (1.2).

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For the remainder of this section, we gather together some definitions and results which we will need in Section 2. First $C([0,\infty),\mathbf{R}^k)$ is the space of continuous mappings from $[0,\infty)$ to $\mathbf{R}^k$), the topology being that of uniform convergence on any compact interval of $[0,\infty)$. If $u \in C([0,\infty),\mathbf{R}^k)$, then for every $m \in N^+$ we define the seminorm $|u|_m$ by

$$|u|_m = \sup_{t\in[0,t_m]} |u(t)|$$

where $t_m \uparrow \infty$ (and $0 < t_1 < t_2 < \cdots$); the metric (since the topology is determined by a countable number of seminorms) is defined by

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{|x-y|_m}{1+|x-y|_m}.$$

Note $C([0,\infty),\mathbf{R}^k)$ is a Fréchet space. $BC([0,\infty),\mathbf{R}^k)$ will denote the space of bounded continuous mappings from $[0,\infty)$ to $\mathbf{R}^k$. If $u \in BC([0,\infty),\mathbf{R}^k)$, then we write

$$|u|_\infty = \sup_{t\in[0,\infty)} |u(t)|.$$

A sequence (x_n) in $C([0,\infty),\mathbf{R}^k)$ is said to converge to $x \in C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$, written $x_n \to x$ as $n \to \infty$ in $C([0,\infty),\mathbf{R}^k)$, if $|x_n - x|_m \to 0$ as $n \to \infty$ for every $m \in N^+$. A map $F : C([0,\infty),\mathbf{R}^k) \to C([0,\infty),\mathbf{R}^k)$ is said to be continuous (and we say F is c-continuous) if

$$x_n \to x \text{ in } C\left([0,\infty),\mathbf{R}^k\right) \text{ implies } F x_n \to F x \text{ in } C\left([0,\infty),\mathbf{R}^k\right).$$

A sequence (x_n) in $BC([0,\infty),\mathbf{R}^k)$ is said to converge strictly (see [5,6,8]) to $x \in BC([0,\infty),\mathbf{R}^k)$, written $x_n \rightharpoonup x$ as $n \to \infty$, if $|x_n|_\infty = \sup_{t\in[0,\infty)} |x_n(t)|$ is uniformly bounded and $|x_n - x|_m \to 0$ as $n \to \infty$ for every $m \in N^+$. A map $F : BC([0,\infty),\mathbf{R}^k) \to BC([0,\infty),\mathbf{R}^k)$ is said to be s-continuous if

$$x_n \rightharpoonup x \quad \text{implies} \quad F x_n \rightharpoonup F x.$$

Recall [9] the Arzela-Ascoli theorem says that a set $\Omega \subseteq C([0,\infty),\mathbf{R}^k)$ is relatively compact iff Ω is uniformly bounded and equicontinuous on each compact interval of $[0,\infty)$.

Next, we state and prove a result which will be used frequently in Section 2.

THEOREM 1.1. *Let $(z_n)_{n\in N^+}$ be bounded and equicontinuous on each compact subset of $[0,\infty)$ and let $z_n(t) \to z(t)$, as $n \to \infty$, for each $t \in [0,\infty)$. Then $z_n \to z$ in $C([0,\infty),\mathbf{R}^k)$ (i.e., $|z_n - z|_m \to 0$ as $n \to \infty$ for each $m \in N^+$).*

PROOF. The proof can be found in [5, p. 4]. For completeness, we provide the details here. Suppose $z_n \not\to z$ in $C([0,\infty),\mathbf{R}^k)$. Then there exists $m \in N^+$ with $z_n \not\to z$ in $C([0,t_m],\mathbf{R}^k)$. As a result there exists $\epsilon > 0$, a subsequence S of N^+, with

$$|z_n - z|_m \geq \epsilon, \qquad \text{for all } n \in S. \tag{1.4}$$

Now since $\{z_n\}_{n\in N^+}$ is a compact subset of $C([0,\infty),\mathbf{R}^k)$ then there exists $y \in C([0,\infty),\mathbf{R}^k)$ and a subsequence S_1 of S with $z_n \to y$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$ in S_1. However, (1.4) implies $|y - z|_m \geq \epsilon$. This is a contradiction. ∎

Finally, we state the fixed point theorem [1–4] which will be used throughout Section 2.

THEOREM 1.2. *Let C be a complete, convex subset of a metrizable (with metric d) locally convex linear topological space E with Q a closed, convex, proper subset of C, $0 \in Q$, and*

$$U_i = \left\{ x \in E : \ d(x,Q) < \frac{1}{i} \right\} \subseteq C$$

for i sufficiently large. Assume $F : Q \to C$ is a continuous, compact map. In addition, suppose

$$\begin{gathered} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x = \lambda F(x) \text{ and } 0 \leq \lambda < 1, \text{ then there exists } j_0 \in \{1,2,\dots\} \\ \text{with } \lambda_j F(x_j) \in Q, \text{ for each } j \geq j_0 \end{gathered} \tag{1.5}$$

holds. Then F has a fixed point in Q.

2. EXISTENCE AND APPROXIMATION

We begin this section by establishing an existence and approximation principle for the operator equation

$$y(t) = F\,y(t), \qquad \text{on } [0,\infty). \tag{2.1}$$

Associated with (2.1), we consider for each $n \in N^+$ (think of these as corresponding numerical approximations), the equation

$$y(t) = F_n\,y(t), \qquad \text{on } [0,\infty). \tag*{(2.2)^n}$$

A collection

$$\mathcal{K} = \{T_\alpha : \ \alpha \in J \ \text{ (some index set)}\}\ ,$$

where $T_\alpha : X \to C([0,\infty), \mathbf{R}^k)$ for each $\alpha \in J$, is collectively compact in $C([0,\infty), \mathbf{R}^k)$ if for each bounded set Ω of X the set $\mathcal{K}\,\Omega$ is relatively compact in $C([0,\infty), \mathbf{R}^k)$; here X is a subset of $C([0,\infty), \mathbf{R}^k)$.

THEOREM 2.1. *Let Q be a closed, bounded, convex subset of $C([0,\infty), \mathbf{R}^k)$ with $0 \in Q$. Assume the following conditions are satisfied:*

$$\text{for each } n \in N^+,\ F_n : Q \to C\left([0,\infty), \mathbf{R}^k\right) \ \text{ is c-continuous,} \tag{2.3}$$

$$F : Q \to C\left([0,\infty), \mathbf{R}^k\right) \ \text{ is c-continuous,} \tag{2.4}$$

$$\mathcal{K} = \left\{F_n : \ n \in N^+\right\} \ \text{ is collectively compact,} \tag{2.5}$$

$$\begin{gathered} \text{for each } n \in N^+, \ \text{ if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{with } x_j \to x \left(\text{in } C\left([0,\infty), \mathbf{R}^k\right)\right) \text{ and } \lambda_j \to \lambda \text{ and if } x = \lambda\,F_n\,(x) \\ \text{with } 0 \le \lambda < 1, \ \text{ then } \lambda_j\,F_n(x_j) \in Q \text{ for } j \text{ sufficiently large,} \end{gathered} \tag{2.6}$$

and

$$\begin{gathered} \text{for each } m \in N^+, |F_n\,y - F\,y|_m \to 0 \text{ as } n \to \infty, \\ \text{uniformly for } y \in Q. \end{gathered} \tag{2.7}$$

Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of (2.2)^n, $n \in S$, with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,\infty), \mathbf{R}^k)$ and x_0 is a solution of (2.1) in Q.

REMARK 2.1. Notice if Q and $F(Q)$ are bounded subsets of $BC([0,\infty), \mathbf{R}^k)$, then (2.4) implies $F : Q \to BC([0,\infty), \mathbf{R}^k)$ is s-continuous. This is the situation we usually encounter in applications.

PROOF. For each $n \in N^+$, F_n has a fixed point in Q (apply Theorem 1.2 with $C = E = C([0,\infty), \mathbf{R}^k)$; note $F_n(Q)$ is relatively compact in $C([0,\infty), \mathbf{R}^k)$ by (2.5)). Thus, there exists $x_n \in Q$ with $x_n = F_n\,x_n$. Let

$$\Omega = \overline{\{F_n\,y : \ y \in Q,\ n = 1, 2, \dots\}} \ \left(\text{closure in } C\left([0,\infty), \mathbf{R}^k\right)\right).$$

Now Ω is a compact subset of $C([0,\infty), \mathbf{R}^k)$ (see (2.5)) so there exists a subsequence of (x_n) (without loss of generality assume its the whole sequence) with $x_n \to x_0$ in $C([0,\infty), \mathbf{R}^k)$.

We now claim $F_n\,x_n \to F\,x_0$ in $C([0,\infty), \mathbf{R}^k)$ as $n \to \infty$. To see this fix $m \in N^+$ and let $t \in [0, t_m]$. Then

$$|F_n\,x_n(t) - F\,x_0(t)| \le |F_n\,x_n(t) - F\,x_n(t)| + |F\,x_n(t) - F\,x_0(t)|,$$

so

$$|F_n\,x_n - F\,x_0|_m \le \sup_{y \in Q} |F_n\,y - F\,y|_m + |F\,x_n - F\,x_0|_m. \tag{2.8}$$

Thus (2.4), (2.7), and (2.8) imply that our claim is true i.e., $F_n x_n \to F x_0$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$. Now $x_n = F_n x_n$, $x_n \to x_0$ in $C([0,\infty),\mathbf{R}^k)$ and $F_n x_n \to F x_0$ in $C([0,\infty),\mathbf{R}^k)$ imply $x_0 = F x_0$. ∎

REMARK 2.2. Notice (2.6) can be replaced by any condition that guarantees that F_n has a fixed point in Q for each $n \in N^+$; also Q need only be a closed, bounded subset of $C([0,\infty),\mathbf{R}^k)$.

REMARK 2.3. In our applications Q is usually a convex, bounded subset of $BC([0,\infty),\mathbf{R}^k)$.

From an application viewpoint it is of interest to put conditions (which are reasonable and easy to check) on F_n so that (2.6) is automatically satisfied. We present one such result in the next theorem.

THEOREM 2.2. *Suppose*

$$\begin{gathered}\text{for each } n \in N^+, \text{ there exists a constant } \delta > 0 \text{ (independent of } n\text{) and}\\ \text{a continuous function } \psi : [0,\infty) \to (\delta,\infty) \text{ (independent of } n\text{) with}\\ |u(t)| \le \psi(t) - \delta,\ t \in [0,\infty), \text{ for any solution } u \in C\left([0,\infty),\mathbf{R}^k\right)\\ \text{which satisfies } u = \lambda F_n u, \text{ for } 0 \le \lambda < 1\end{gathered} \tag{2.9}$$

and

$$\begin{gathered}\text{for each } n \in N^+, \text{ there exists } q_0 > 0 \text{ (which may depend on } n\text{)}\\ \text{with } |F_n v(t)| \le \psi(t),\ t \in [q_0,\infty), \text{ for any } v \in C\left([0,\infty),\mathbf{R}^k\right)\\ \text{which satisfies } |v(t)| \le \psi(t), \text{ for } t \in [0,\infty)\end{gathered} \tag{2.10}$$

are satisfied. Let

$$Q = \left\{y \in C\left([0,\infty),\mathbf{R}^k\right) : |y(t)| \le \psi(t),\ \text{for } t \in [0,\infty)\right\}$$

and assume (2.3), (2.4), (2.5), *and* (2.7) *hold. Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of* (2.2)^n, $n \in S$, *with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,\infty),\mathbf{R}^k)$ and x_0 is a solution of* (2.1) *in Q.*

PROOF. First notice Q is a nonempty, closed, convex, bounded subset of $C([0,\infty),\mathbf{R}^k)$. The result follows immediately from Theorem 2.1 once we show (2.6) is satisfied. Fix $n \in N^+$. Take a sequence $\{(y_j,\lambda_j)\}_{j=1}^{\infty}$ in $\partial Q \times [0,1]$ with $\lambda_j \to \lambda$ and $y_j \to y$ in $C([0,\infty),\mathbf{R}^k)$ with $y = \lambda F_n y$ and $0 \le \lambda < 1$. We must show $\lambda_j F_n(y_j) \in Q$ for j sufficiently large. First notice $y_j \in Q$ together with (2.10) implies that there exists $q_0 > 0$ with $|F_n y_j(t)| \le \psi(t)$ for $t \in [q_0,\infty)$ and $j \in \{1,2,\dots\}$. Thus, we have

$$|\lambda_j F_n y_j(t)| \le \psi(t), \qquad \text{for } t \in [q_0,\infty) \text{ and } j \in \{1,2,\dots\}. \tag{2.11}$$

Next let $t \in [0,q_0]$. Since $F_n : Q \to C([0,\infty),\mathbf{R}^k)$ is c-continuous we know $F_n y_j \to F_n y$ (as $j \to \infty$) uniformly on $[0,q_0]$. In addition, since $\lambda_j \to \lambda$ and $F_n(Q)$ is relatively compact in $C([0,\infty),\mathbf{R}^k)$ (see (2.5)), we have $\lambda_j F_n y_j \to \lambda F_n y$ (as $j \to \infty$) uniformly on $[0,q_0]$. Thus for $\delta > 0$ (here δ is as in (2.9)) there exists $j_0 \in \{1,2,\dots\}$ with

$$|\lambda_j F_n y_j(t)| \le |\lambda F_n y(t)| + \delta, \qquad \text{for } t \in [0,q_0] \text{ and } j \ge j_0. \tag{2.12}$$

Now $y = \lambda F_n y$ so (2.9) implies

$$|\lambda F_n y(t)| \le \psi(t) - \delta,$$

and this together with (2.12) implies for $j \ge j_0$ that

$$|\lambda_j F_n y_j(t)| \le \psi(t), \qquad \text{for } t \in [0,q_0]. \tag{2.13}$$

Now (2.11) and (2.13) guarantee that $\lambda_j F_n y_j \in Q$ for $j \geq j_0$. The result now follows from Theorem 2.1. ∎

REMARK 2.4. Notice (2.9) and (2.10) can be replaced by any condition that guarantees

(i) F_n has a fixed point y_n for each $n \in N^+$ and
(ii) there exists a continuous function $\psi : [0,\infty) \to (0,\infty)$ with $|y_n(t)| \leq \psi(t)$, $t \in [0,\infty)$ for each $n \in N^+$ (in this situation Q will be as in the statement of Theorem 2.2).

REMARK 2.5. If $M > 0$ is a constant and $\psi(t) = M$, for all $t \in [0,\infty)$, then in (2.9) we may replace $u \in C([0,\infty), \mathbf{R}^k)$ with $u \in BC([0,\infty), \mathbf{R}^k)$ since Q in this case is a closed, bounded set in $BC([0,\infty), \mathbf{R}^k)$.

We now consider a special case of (2.1), namely the nonlinear integral equation

$$y(t) = h(t) + \int_0^\infty \kappa(t,s,y(s))\,ds, \qquad \text{for } t \in [0,\infty). \tag{2.14}$$

Finite section approximations for (2.14) are given by

$$y(t) = h(t) + \int_0^{t_n} \kappa(t,s,y(s))\,ds, \qquad \text{for } t \in [0,\infty), \tag*{$(2.15)^n$}$$

where $t_n \uparrow \infty$ (and $0 < t_1 < t_2 < \cdots$) and $n \in N^+$.

THEOREM 2.3. *Assume the following conditions are satisfied:*

$$h \in BC\left([0,\infty), \mathbf{R}^k\right), \tag{2.16}$$

$$\begin{gathered} \textit{for each } t \in [0,\infty), \textit{ the map } s \mapsto \kappa_t(s,u) \textit{ is measurable} \\ \textit{for all } u \in \mathbf{R}^k\,; \textit{ here } \kappa_t(s,u) = \kappa(t,s,u), \end{gathered} \tag{2.17}$$

$$\begin{gathered} \textit{for each } t \in [0,\infty), \textit{ the map } u \mapsto \kappa_t(s,u) \\ \textit{is continuous for almost all } s \in [0,\infty), \end{gathered} \tag{2.18}$$

$$\Phi_b = \sup_{t\in[0,\infty)} \int_0^\infty \sup_{|u|\leq b} |\kappa(t,s,u)|\,ds < \infty, \textit{ for each } b > 0, \tag{2.19}$$

$$\begin{gathered} \Gamma_b(t',t) = \int_0^\infty \sup_{|u|\leq b} |\kappa(t',s,u) - \kappa(t,s,u)|\,ds \to 0 \\ \textit{as } t' \to t, \textit{ for each } t \in [0,\infty), \textit{ for each } b > 0, \end{gathered} \tag{2.20}$$

$$\lim_{t\to\infty} \int_0^\infty \sup_{|u|\leq b} |\kappa(t,s,u)|\,ds = 0, \textit{ for each } b > 0 \tag{2.21}$$

and

$$\begin{gathered} \textit{for each } n \in N^+, \textit{ there exists a constant } M_0 > \sup_{[0,\infty)} |h(t)| = |h|_\infty \\ \textit{with } |u(t)| \leq M_0,\, t \in [0,\infty), \textit{ for any function } u \in BC\left([0,\infty), \mathbf{R}^k\right), \\ \textit{which satisfies } u(t) = \lambda\left(h(t) + \int_0^{t_n} \kappa(t,s,u(s))\,ds\right), \textit{ for } 0 \leq \lambda < 1. \end{gathered} \tag{2.22}$$

Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of $(2.15)^n$, $n \in S$, with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,\infty), \mathbf{R}^k)$ and x_0 is a solution of (2.14).

PROOF. Let

$$F\,y(t) = h(t) + \int_0^\infty \kappa(t,s,y(s))\,ds \quad \text{and} \quad F_n\,y(t) = h(t) + \int_0^{t_n} \kappa(t,s,y(s))\,ds, \quad n \in N^+.$$

Notice F, F_n $(n \in N^+) : BC([0,\infty),\mathbf{R}^k) \to BC([0,\infty),\mathbf{R}^k)$. Let

$$Q = \left\{ y \in C\left([0,\infty),\mathbf{R}^k\right) : y \in BC\left([0,\infty),\mathbf{R}^k\right) \text{ and } |y|_\infty = \sup_{[0,\infty)} |y(t)| \le M_0 + 1 = R \right\}.$$

Notice Q is a closed, convex, bounded subset of $C([0,\infty),\mathbf{R}^k)$ (to see that Q is closed let $y_n \in Q$, $n \in N^+$ with $y_n \to y$ in $C([0,\infty),\mathbf{R}^k)$. Fix $t \in [0,\infty)$. Then, since $|y_n(t)| \le R$ for $n \in N^+$, we have immediately, since $y_n \to y$ uniformly on $[0,t+1]$, that $|y(t)| \le R$. Consequently, $y \in Q$). The result will follow from Theorem 2.2 once we show (2.3), (2.4), (2.5), (2.7), (2.9), and (2.10) are satisfied. We first show (2.5) is true. For any $y \in Q$ (note $|y(t)| \le R$ for $t \in [0,\infty)$) and any $n \in N^+$, we have for t', $t \in [0,\infty)$,

$$|F_n y(t)| \le |h(t)| + \int_0^{t_n} |\kappa(t,s,y(s))|\,ds \le |h|_\infty + \Phi_R$$

and

$$\begin{aligned} |F_n y(t') - F_n y(t)| &\le |h(t') - h(t)| + \int_0^{t_n} |\kappa(t',s,y(s)) - \kappa(t,s,y(s))|\,ds \\ &\le |h(t') - h(t)| + \Gamma_R(t',t). \end{aligned}$$

Consequently, $\{F_n : n \in N^+\}$, where $F_n : Q \to C([0,\infty),\mathbf{R}^k)$, is uniformly bounded and equicontinuous on each compact interval of $[0,\infty)$ and so by the Arzela-Ascoli theorem $\{F_n : n \in N^+\}$ is collectively compact i.e., (2.5) is true.

To show (2.3), fix $n \in N^+$ and let $y_j \to y$ in Q (in $C([0,\infty),\mathbf{R}^k)$) as $j \to \infty$ (here $j \in N^+$). Now $|y_j(t)| \le R = M_0 + 1$ for $t \in [0,\infty)$ and $j \in N^+$. Also for each $t \in [0,\infty)$,

$$F_n y_j(t) - F_n y(t) = \int_0^{t_n} [\kappa(t,s,y_j(s)) - \kappa(t,s,y(s))]\,ds.$$

Notice for each $t \in [0,\infty)$,

$$\kappa(t,s,y_j(s)) \to \kappa(t,s,y(s)) \text{ as } j \to \infty, \qquad \text{for a.e. } s \in [0,\infty)$$

and

$$|\kappa(t,s,y_j(s)) - \kappa(t,s,y(s))| \le 2 \sup_{|u|\le R} |\kappa(t,s,u)|.$$

This together with the Lebesgue dominated convergence theorem (see (2.19)) yields

$$F_n y_j(t) \to F_n y(t) \text{ as } j \to \infty, \qquad \text{for each } t \in [0,\infty). \tag{2.23}$$

From above we know $\{F_n y_j : j \in N^+\}$ is bounded and equicontinuous on each compact subinterval of $[0,\infty)$. This together with (2.23) and Theorem 1.1 yields $F_n y_j \to F_n y$ (as $j \to \infty$) in $C([0,\infty),\mathbf{R}^k)$. Thus for each $n \in N^+$, $F_n : Q \to C([0,\infty),\mathbf{R}^k)$ is c-continuous i.e., (2.3) is true. A similar argument shows (2.4) is true (since its also easy to check that $F : Q \to C([0,\infty),\mathbf{R}^k)$ is compact).

To show (2.7), fix $m \in N^+$. Now notice for each $t \in [0,t_m]$, $n \in N^+$, and $y \in Q$, that

$$F y(t) - F_n y(t) = \int_{t_n}^{\infty} \kappa(t,s,y(s))\,ds$$

and so

$$|F y(t) - F_n y(t)| \le \sup_{t\in[0,t_m]} \int_{t_n}^{\infty} \sup_{|u|\le R} |\kappa(t,s,u)|\,ds.$$

This together with (2.19) implies $|Fy - F_n y|_m \to 0$ as $n \to \infty$, uniformly for $y \in Q$ i.e., (2.7) is true. We next show (2.10) is satisfied with $\psi(t) = R = M_0 + 1$. To see this take $n \in N^+$ and $v \in C([0,\infty), \mathbf{R}^k)$ with $|v(t)| \le \psi(t) = R$ for $t \in [0,\infty)$. Then

$$\begin{aligned}|F_n v(t)| &\le |h(t)| + \int_0^{t_n} |\kappa(t,s,v(s))|\,ds \\ &\le |h|_\infty + \int_0^\infty \sup_{|u|\le R} |\kappa(t,s,u)|\,ds.\end{aligned}$$

This together with (2.21) (and the fact that $M_0 > |h|_\infty$) implies that (2.10) is true. Finally, notice (2.22) implies (2.9) is true (see also Remark 2.5) with $\psi(t) = R = M_0 + 1$ and $\delta = 1/2$. The result now follows from Theorem 2.2. ∎

REMARK 2.6. Notice (2.21) and (2.22) can be replaced by any condition that guarantees

(i) that $(2.15)^n$ has a solution y_n for each $n \in N^+$ and

(ii) there exists M_0 with $|y_n(t)| \le M_0$, $t \in [0,\infty)$ for each $n \in N^+$.

A similar remark will apply to Theorem 2.4.

REMARK 2.7. Notice $(2.15)^n$ for fixed $n \in N^+$ determines $y(t)$ for $t > t_n$ in terms of $y(x)$ for $x \in [0, t_n]$ so in fact $(2.15)^n$ reduces to an integral equation on the finite interval $[0, t_n]$. See [10,11] for a discussion of nonlinear integral equations on compact intervals. Notice as well that (2.21) in Theorem 2.3 can be replaced by: there exists $q_0 \in (0,\infty)$ with

$$\sup_{t\in[q_0,\infty)} \left\{ |h(t)| + \int_0^\infty \sup_{|u|\le R} |\kappa(t,s,u)|\,ds \right\} \le R,$$

here $R = M_0 + 1$ where M_0 is given in (2.22).

A special case of (2.14) is the Hammerstein equation

$$y(t) = h(t) + \int_0^\infty k(t,s)\, f(s, y(s))\,ds, \qquad \text{for } t \in [0,\infty). \tag{2.24}$$

Finite section approximations for (2.14) are given by

$$y(t) = h(t) + \int_0^{t_n} k(t,s)\, f(s, y(s))\,ds, \qquad \text{for } t \in [0,\infty), \tag*{$(2.25)^n$}$$

where $t_n \uparrow \infty$ (and $0 < t_1 < t_2 < \cdots$) and $n \in N^+$.

THEOREM 2.4. *Let $1 \le \alpha < \infty$ be a constant and β the conjugate to α. Assume the following conditions are satisfied:*

$$h \in BC\left([0,\infty), \mathbf{R}^k\right), \tag{2.26}$$

$$\text{for each } t \in [0,\infty), \text{ the map } s \mapsto k(t,s) \text{ is measurable,} \tag{2.27}$$

$$\sup_{t\in[0,\infty)} \int_0^\infty |k(t,s)|^\alpha\,ds < \infty, \tag{2.28}$$

$$\int_0^\infty |k(t',s) - k(t,s)|^\alpha\,ds \to 0 \text{ as } t' \to t, \text{ for each } t \in [0,\infty), \tag{2.29}$$

$$\lim_{t\to\infty} \int_0^\infty |k(t,s)|^\alpha\,ds = 0, \tag{2.30}$$

$f : [0,\infty) \times \mathbf{R}^k \to \mathbf{R}^k$ is a L^β-Carathéodory function: by this we mean:

(i) *$s \mapsto f(s,y)$ is measurable, for all $y \in \mathbf{R}^k$*

(ii) *$y \mapsto f(s,y)$ is continuous for a.e. $s \in [0,\infty)$* (2.31)

(iii) *for each $r > 0$ there exists $\tau_r \in L^\beta[0,\infty)$, such that $|y| \le r$ implies $|f(s,y)| \le \tau_r(s)$, for almost all $s \in [0,\infty)$*

and

$$\begin{aligned} &\textit{for each } n \in N^+, \textit{ there exists a constant } M_0 > |h|_\infty \textit{ with} \\ &|u(t)| \le M_0,\ t \in [0,\infty), \textit{ for any function } u \in BC\left([0,\infty),\mathbf{R}^k\right) \textit{ which} \\ &\textit{satisfies } u(t) = \lambda \left(h(t) + \int_0^{t_n} k(t,s)\, f(s,u(s))\, ds \right), \textit{ for } 0 \le \lambda < 1. \end{aligned} \tag{2.32}$$

Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of $(2.25)^n$, $n \in S$, with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,\infty),\mathbf{R}^k)$, and x_0 is a solution of (2.24).

PROOF. Let $\kappa(t,s,u) = k(t,s)\, f(s,u)$ and notice the conditions of Theorem 2.3 are satisfied. ∎

We now discuss the solution sets of (2.1) and $(2.2)^n$, $n \in N^+$. Let A_n ($\subseteq C([0,\infty),\mathbf{R}^k)$) be the solution set of $(2.2)^n$ and A (assuming $A \ne \emptyset$, which is the case if the assumptions of Theorem 2.1 are satisfied) the solution set of (2.1). We say $A_n \to A$ in $C([0,\infty),\mathbf{R}^k)$, as $n \to \infty$ in N^+, if for each $m \in \{1,2,\dots\}$, $A_n \to A$ in $C([0,t_m],\mathbf{R}^k)$ as $n \to \infty$ in N^+. Now fix m. Note $A_n \to A$ in $C([0,t_m],\mathbf{R}^k)$ as $n \to \infty$ in N^+ if for every $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that A_n lies in an ϵ-neighborhood of A for $n \ge n(\epsilon)$. If we assume

$$K = \left\{A_n : n \in N^+\right\} \text{ is relatively compact } \left(\text{in } C\left([0,\infty),\mathbf{R}^k\right)\right) \tag{2.33}$$

and

$$K' \subseteq A \tag{2.34}$$

hold, then $A_n \to A$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$ in N^+; here K' is the set of cluster points of K.

REMARK 2.8. Our defination of a cluster point of K is as follows: x is a cluster point of $K = \{A_n : n \in N^+\}$ if there exists a subsequence S of N^+ and $x_n \in A_n$, $n \in S$ with $x_n \to x$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$ in S.

To see this suppose $A_n \not\to A$ in $C([0,\infty),\mathbf{R}^k)$. Then there exists $m \in N^+$ with $A_n \not\to A$ in $C([0,t_m],\mathbf{R}^k)$. Thus there exists $\epsilon > 0$, a subsequence P_1 of N^+ and $x_n \in A_n$ with

$$|x_n - x|_m \ge \epsilon, \qquad \text{for all } n \in P_1 \text{ and for all } x \in A. \tag{2.35}$$

Also notice (2.33) implies that there exists $y \in C([0,\infty),\mathbf{R}^k)$ and a subsequence P_2 of P_1 with $x_n \to y$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$ in P_2. By definition $y \in K'$. Also (2.35) implies

$$|y - x|_m \ge \epsilon, \qquad \text{for all } x \in A.$$

This contradicts (2.34).

REMARK 2.9. Suppose the conditions of Theorem 2.1 are satisfied. Let A_n be the solution set of $(2.2)^n$ in Q and A the solution set of (2.1) in Q. Notice (2.34) is automatically satisfied. To see this let $x \in K'$. Then there exists a subsequence P_3 of N^+ and $x_n \in A_n$ (i.e., $x_n = F_n x_n$) with $x_n \to x$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$ in P_3. Thus for each $m \in N^+$,

$$|F_n x_n - F x|_m \le \sup_{y \in Q} |F_n y - F y|_m + |F x_n - F x|_m .$$

Now (2.4) and (2.7) imply $|F_n x_n - F x|_m \to 0$ as $n \to \infty$ in P_3 i.e., $|x_n - F x|_m \to 0$ as $n \to \infty$ in P_3. We can do this for each $m \in N^+$, so $x_n \to F x$ in $C([0,\infty),\mathbf{R}^k)$ as $n \to \infty$ in P_3. Thus $x = F x$ i.e., $x \in A$.

It is of interest now to study numerical integration approximations of $(2.15)^n$. However before we do so, we first discuss operator equations on compact intervals. We will just sketch the proofs, since the ideas will be similar to those in Theorem 2.1. In particular consider

$$y(t) = F y(t), \qquad \text{on } [0,T] \tag{2.36}$$

and the associated family of problems

$$y(t) = F_n y(t), \qquad \text{on } [0,T], \tag{2.37^n}$$

here $n \in N^+$. A collection

$$\mathcal{K} = \{T_\alpha : \ \alpha \in J \ \ \text{(some index set)}\} ,$$

where $T_\alpha : X \to C([0,T],\mathbf{R}^k)$, for each $\alpha \in J$, is collectively compact in $C([0,T],\mathbf{R}^k)$ if for each bounded set Ω of X the set $\mathcal{K}\,\Omega$ is relatively compact in $C([0,T],\mathbf{R}^k)$; here X is a subset of $C([0,T],\mathbf{R}^k)$.

THEOREM 2.5. *Suppose*

$$\begin{gathered}\text{for each } n \in N^+, \text{ there is a constant } M, \text{ independent of } \lambda \text{ and } n,\\ \text{with } |y|_0 = \sup_{[0,T]} |y(t)| \neq M, \text{ for any solution } y \in C([0,T],\mathbf{R}^k)\\ \text{to } y(t) = \lambda\, F_n\, y(t) \text{ on } [0,T], \text{ for each } \lambda \in (0,1)\end{gathered} \tag{2.38}$$

holds. Let

$$U = \left\{u \in C\left([0,T],\mathbf{R}^k\right) : \ |u|_0 < M\right\}$$

and assume the following conditions are satisfied:

$$\text{for each } n \in N^+,\ F_n : \overline{U} \to C\left([0,T],\mathbf{R}^k\right) \ \text{is continuous,} \tag{2.39}$$

$$F : \overline{U} \to C\left([0,T],\mathbf{R}^k\right) \ \text{is continuous,} \tag{2.40}$$

$$\mathcal{K} = \{F_n : \ n \in N^+\} \ \text{is collectively compact,} \tag{2.41}$$

and

$$|F_n\, y - F\, y|_0 \to 0 \ \text{as} \ n \to \infty, \ \text{uniformly for } y \in \overline{U}. \tag{2.42}$$

Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of (2.37)^n, $n \in S$, with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,T],\mathbf{R}^k)$ and x_0 is a solution of (2.36).

PROOF. Using the Leray Schauder alternative [4,12] we see that (2.37)^n has a solution $x_n \in \overline{U}$ (i.e $x_n = F_n\, x_n$ and $x_n \in \overline{U}$), for each $n \in N^+$. Let

$$\Omega = \overline{\{F_n\, y : \ y \in \overline{U},\ n = 1,2,\dots\}} \ \left(\text{closure in } C\left([0,T],\mathbf{R}^k\right)\right).$$

There exists a subsequence of (x_n) (without loss of generality assume its the whole sequence) with $x_n \to x_0$ in $C([0,T],\mathbf{R}^k)$. Also notice

$$|F_n\, x_n - F\, x_0|_0 \le \sup_{y \in \overline{U}} |F_n\, y - F\, y|_0 + |F\, x_n - F\, x_0|_0 ,$$

so $F_n\, x_n \to F\, x_0$ in $C([0,T],\mathbf{R}^k)$. Thus $x_0 = F\, x_0$. ∎

REMARK 2.10. There is an obvious analogue of Theorem 2.5 for solutions in $L^p([0,T],\mathbf{R}^k)$, $1 \le p < \infty$ (we leave the statement and proof to the reader).

REMARK 2.11. It is also possible to discuss the solution set A_n of (2.37)^n and A of (2.36). Under reasonable assumptions we can again show $A_n \to A$ in $C([0,T],\mathbf{R}^k)$ (the definition of convergence in this case is obvious). We leave the details to the reader.

We now consider as a special case of (2.36), the nonlinear integral equation

$$y(t) = h(t) + \int_0^T \kappa(t,s,y(s))\, ds, \qquad \text{for } \ t \in [0,T]. \tag{2.43}$$

The associated numerical integration approximations of (2.43) will be

$$y(t) = h(t) + \sum_{i=0}^{n} \kappa^\star(t, s_{i\,n}, y(s_{i\,n})), \qquad \text{for } \ t \in [0,T] \tag{2.44^n}$$

for $n \in N^+$; here $0 \le s_{i\,n} \le T$.

THEOREM 2.6. *Suppose*

$$\begin{gathered}\text{for each } n \in N^+, \text{ there is a constant } M, \text{ independent of } \lambda \text{ and } n,\\ \text{with } |y|_0 = \sup_{[0,T]} |y(t)| \neq M, \text{ for any solution } y \in C([0,T],\mathbf{R}^k) \text{ to}\\ y(t) = \lambda \left[h(t) + \sum_{i=0}^{n} \kappa^\star(t, s_{in}, y(s_{in})) \right] \text{ on } [0,T], \text{ for each } \lambda \in (0,1)\end{gathered} \tag{2.45}$$

holds. Let

$$U = \left\{ u \in C\left([0,T],\mathbf{R}^k\right) : \ |u|_0 < M \right\}$$

and assume the following conditions are satisfied:

$$h \in C\left([0,T],\mathbf{R}^k\right), \tag{2.46}$$

$$\begin{gathered}\text{for each } t \in [0,T], \text{ the map } s \mapsto \kappa_t(s,u) \text{ is measurable}\\ \text{for all } u \in \mathbf{R}^k; \text{ here } \kappa_t(s,u) = \kappa(t,s,u),\end{gathered} \tag{2.47}$$

$$\begin{gathered}\text{for each } t \in [0,T], \text{ the map } u \mapsto \kappa_t(s,u)\\ \text{is continuous for almost all } s \in [0,T],\end{gathered} \tag{2.48}$$

$$\Phi_b = \sup_{t\in[0,T]} \int_0^T \sup_{|u|\le b} |\kappa(t,s,u)|\, ds < \infty, \text{ for each } b > 0, \tag{2.49}$$

$$\begin{gathered}\Gamma_b(t',t) = \int_0^T \sup_{|u|\le b} |\kappa(t',s,u) - \kappa(t,s,u)|\, ds \to 0\\ \text{as } t' \to t, \text{ for each } t \in [0,T], \text{ for each } b > 0,\end{gathered} \tag{2.50}$$

$$\begin{gathered}\text{for each } t \in [0,T], \text{ the map } u \mapsto \kappa_t^\star(s,u) \text{ is continuous}\\ \text{for all } s \in [0,T]; \text{ here } \kappa_t^\star(s,u) = \kappa^\star(t,s,u),\end{gathered} \tag{2.51}$$

$$\Phi_b^\star = \sup_{t\in[0,T]} \sup_{n\in N^+} \sum_{i=0}^{n} \sup_{|u|\le b} |\kappa^\star(t,s_{in},u)| < \infty, \text{ for each } b > 0, \tag{2.52}$$

$$\begin{gathered}\Gamma_b^\star(t',t) = \sup_{n\in N^+} \sum_{i=0}^{n} \sup_{|u|\le b} |\kappa^\star(t',s_{in},u) - \kappa^\star(t,s_{in},u)| \to 0\\ \text{as } t' \to t, \text{ for each } t \in [0,T], \text{ for each } b > 0,\end{gathered} \tag{2.53}$$

and

$$\begin{gathered}\int_0^T \kappa(t,s,y(s))\, ds - \sum_{i=0}^{n} \kappa^\star(t,s_{in},y(s_{in}))| \to 0 \text{ as } n \to \infty,\\ \text{uniformly for } y \in \overline{U}.\end{gathered} \tag{2.54}$$

Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of $(2.44)^n$, $n \in S$, with $x_n \to x_0$ (as $n \to \infty$ in S) in $C([0,T],\mathbf{R}^k)$, and x_0 is a solution of (2.43).

PROOF. Let

$$F\,y(t) = h(t) + \int_0^T \kappa(t,s,y(s))\, ds$$

and

$$F_n\,y(t) = h(t) + \sum_{i=0}^{n} \kappa^\star(t,s_{in},y(s_{in})).$$

Notice F, $F_n\,(n \in N^+) : C([0,T],\mathbf{R}^k) \to C([0,T],\mathbf{R}^k)$. Also it is easy to see (see the ideas in Theorem 2.3) that (2.40) is satisfied i.e., $F : \overline{U} \to C([0,T],\mathbf{R}^k)$ is continuous (note it is easy to check using (2.49) and (2.50) that $F : \overline{U} \to C([0,T],\mathbf{R}^k)$ is compact). We now show (2.41) is true. Note for any $y \in \overline{U}$ and any $n \in N^+$ we have for $t,\ t' \in [0,T]$,

$$|F_n\,y(t)| \le |h|_0 + \sum_{i=0}^{n} |\kappa^\star(t, s_{in}, y(s_{in}))| \le |h|_0 + \Phi_M^\star$$

and

$$|F_n\,y(t') - F_n\,y(t)| \le |h(t') - h(t)| + \Gamma_M^\star\,(t',t)\,.$$

Thus $\mathcal{K} = \{F_n : n \in N^+\}$, where $F_n : \overline{U} \to C([0,T],\mathbf{R}^k)$, is uniformly bounded and equicontinuous. Now (2.41) follows from the Arzela-Ascoli theorem. To show (2.40), fix $n \in N^+$, and let $y_j \to y$ in $\overline{U}$ as $j \to \infty$ (here $j \in N^+$). For each $t \in [0,T]$,

$$\kappa^\star(t, s_{in}, y_j(s_{in})) \to \kappa^\star(t, s_{in}, y(s_{in})) \text{ as } j \to \infty$$

and

$$|\kappa^\star(t, s_{in}, y_j(s_{in})) - \kappa^\star(t, s_{in}, y(s_{in}))| \le 2 \sup_{|u| \le M} |\kappa^\star(t, s_{in}, u)|.$$

This together with the Lebesgue dominated convergence theorem yields

$$F_n\,y_j(t) \to F_n\,y(t) \text{ as } j \to \infty, \qquad \text{for each } t \in [0,T]. \tag{2.55}$$

From above we know $\{F_n\,y_j : j \in N^+\}$ is bounded and equicontinuous. This together with (2.55) and Theorem 1.1 yields $F_n\,y_j(t) \to F_n\,y(t)$ (as $j \to \infty$) in $C([0,T],\mathbf{R}^k)$. Thus (2.40) is satisfied. Also (2.42) follows immediately from (2.54). The result now follows from Theorem 2.5. ∎

Lets return and discuss $(2.15)^n$, the finite section approximations of (2.14). Consider, in general, the problem

$$y(t) = h(t) + \int_0^T \kappa(t, s, y(s))\,ds, \qquad \text{for } \ t \in [0,\infty). \tag{2.56}$$

Since (2.56) determines $y(s)$ for $s > T$ in terms of $y(x)$ for $x \in [0,T]$ then (2.56) reduces to an integral equation on $[0,T]$. Hence the results of Theorem 2.6 may be used. Alternatively, we may use Theorem 2.2 (and the ideas in Theorem 2.3); we describe this procedure here. The associated numerical approximations of (2.56) will be

$$y(t) = h(t) + \sum_{i=0}^{n} \kappa^\star(t, s_{in}, y(s_{in})), \qquad \text{for } \ t \in [0,\infty) \tag*{$(2.57)^n$}$$

for $n \in N^+$; here $0 \le s_{in} \le T$.

THEOREM 2.7. *Suppose the following conditions are satisfied:*

$$h \in BC\left([0,\infty), \mathbf{R}^k\right), \tag{2.58}$$

$$\begin{gathered} \textit{for each } t \in [0,\infty), \textit{ the map } s \mapsto \kappa_t(s,u) \textit{ is measurable} \\ \textit{for all } u \in \mathbf{R}^k;\ \textit{here } \kappa_t(s,u) = \kappa(t,s,u), \end{gathered} \tag{2.59}$$

$$\begin{gathered} \textit{for each } t \in [0,\infty), \textit{ the map } u \mapsto \kappa_t(s,u) \\ \textit{is continuous for almost all } s \in [0,T], \end{gathered} \tag{2.60}$$

$$\Phi_b = \sup_{t\in[0,\infty)} \int_0^T \sup_{|u|\le b} |\kappa(t,s,u)|\,ds < \infty, \text{ for each } b>0, \tag{2.61}$$

$$\begin{gathered}\Gamma_b(t',t) = \int_0^T \sup_{|u|\le b} |\kappa(t',s,u)-\kappa(t,s,u)|\,ds \to 0 \\ \text{as } t'\to t \text{ for each } t\in[0,\infty), \text{ for each } b>0,\end{gathered} \tag{2.62}$$

$$\begin{gathered}\text{for each } t\in[0,\infty), \text{ the map } u\mapsto \kappa_t^\star(s,u) \text{ is continuous} \\ \text{for all } s\in[0,T]\,; \text{ here } \kappa_t^\star(s,u)=\kappa^\star(t,s,u),\end{gathered} \tag{2.63}$$

$$\Phi_b^\star = \sup_{t\in[0,\infty)} \sup_{n\in N^+} \sum_{i=0}^{n} \sup_{|u|\le b} |\kappa^\star(t,s_{in},u)| < \infty, \text{ for each } b>0, \tag{2.64}$$

$$\begin{gathered}\Gamma_b^\star(t',t) = \sup_{n\in N^+} \sum_{i=0}^{n} \sup_{|u|\le b} |\kappa^\star(t',s_{in},u)-\kappa^\star(t,s_{in},u)| \to 0 \\ \text{as } t'\to t, \text{ for each } t\in[0,\infty), \text{ for each } b>0,\end{gathered} \tag{2.65}$$

$$\lim_{t\to\infty} \sum_{i=0}^{n} \sup_{|u|\le b} |\kappa^\star(t,s_{in},u)| = 0, \text{ for each } n\in N^+, \text{ for each } b>0, \tag{2.66}$$

and

$$\begin{gathered}\text{for each } n\in N^+, \text{ there is a constant } M_0 > |h|_\infty \text{ with } |u(t)|\le M_0, \\ t\in[0,\infty), \text{ for any function } u\in BC\left([0,\infty),\mathbf{R}^k\right) \text{ which satisfies} \\ u(t) = \lambda\,[h(t) + \sum_{i=0}^{n} \kappa^\star(t,s_{in},u(s_{in}))], \text{ for } 0\le\lambda<1.\end{gathered} \tag{2.67}$$

Let

$$Q = \left\{y\in C\left([0,\infty),\mathbf{R}^k\right) : y\in BC\left([0,\infty),\mathbf{R}^k\right) \text{ and } |y|_\infty \le M_0+1 \equiv R\right\}$$

and assume

$$\begin{gathered}\text{for each } m\in N^+, \left|\int_0^T \kappa(t,s,y(s))\,ds - \sum_{i=0}^{n} \kappa^\star(t,s_{in},y(s_{in}))\right|_m \to 0 \\ \text{as } n\to\infty, \text{ uniformly for } y\in Q\end{gathered} \tag{2.68}$$

holds. Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of (2.57)^n, $n\in S$, with $x_n\to x_0$ (as $n\to\infty$ in S) in $C([0,\infty),\mathbf{R}^k)$, and x_0 is a solution of (2.56).

PROOF. Let

$$F\,y(t) = h(t) + \int_0^T \kappa(t,s,y(s))\,ds$$

and

$$F_n\,y(t) = h(t) + \sum_{i=0}^{n} \kappa^\star(t,s_{in},y(s_{in})).$$

We will apply Theorem 2.2. Notice the ideas in Theorem 2.3 and Theorem 2.6 imply that (2.3), (2.4), and (2.5) are satisfied. In addition, (2.68) implies that (2.7) hold. It remains to show that (2.9) and (2.10) (with $\psi(t)=M_0+1=R$ and $\delta=1/2$) are true. To see (2.10), take $n\in N^+$ and $v\in C([0,\infty),\mathbf{R}^k)$ with $|v(t)|\le\psi(t)=R$ for $t\in[0,\infty)$. Then

$$|F_n\,v(t)| \le |h|_\infty + \sum_{i=0}^{n} \sup_{|u|\le R} |\kappa^\star(t,s_{in},u)|\,.$$

This together with (2.66) implies that (2.10) is satisfied. Finally, notice (2.67) implies (2.9) is true. The result now follows from Theorem 2.2. ∎

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